Sending a Bivariate Gaussian Source over a Gaussian MAC with Feedback

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Abstract

We study the power-versus-distortion trade-off for the transmission of a memoryless bivariate Gaussian source over a two-to-one Gaussian multiple-access channel with perfect causal feedback. In this problem, each of two separate transmitters observes a different component of a memoryless bivariate Gaussian source as well as the feedback from the channel output of the previous time-instants. Based on the observed source sequence and the feedback, each transmitter then describes its source component to the common receiver via an average-power constrained Gaussian multiple-access channel. From the resulting channel output, the receiver wishes to reconstruct both source components with the least possible expected squared-error distortion. We study the set of distortion pairs that can be achieved by the receiver on the two source components.

We present sufficient conditions and necessary conditions for the achievability of a distortion pair. These conditions are expressed in terms of the source correlation and of the signal-to-noise ratio (SNR) of the channel. In several cases the necessary conditions and sufficient conditions coincide. This allows us to show that if the channel SNR is below a certain threshold, then an uncoded transmission scheme that ignores the feedback is optimal. Thus, below this SNR-threshold feedback is useless. We also derive the precise high-SNR asymptotics of optimal schemes.

1 Introduction

This is a sequel to the work in [1] where a bivariate Gaussian source is to be transmitted over a Gaussian multiple-access channel. The new element here is the presence of perfect causal feedback from the channel output to each of the transmitters. As in [1], our interest is in the power-versus-distortion trade-off.

Our setup consists of a memoryless bivariate Gaussian source and a two-to-one Gaussian multiple-access channel with perfect causal feedback. Each of the two transmitters in the multiple-access channel observes a different component of the source as well as feedback from the previous channel outputs. Based on the feedback and the observed source sequence, each transmitter then describes its source component to the common receiver via an average-power constrained Gaussian multiple-access channel. From the output of the channel, the receiver wishes to reconstruct both source components with the least possible expected squared-error distortion. Our interest is in

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characterizing the pairs of squared-error distortions that can be achieved simultaneously on the two source components.

We present sufficient conditions and necessary conditions for the achievability of a distortion pair. These conditions are expressed in terms of the source correlation and the signal-to-noise ratio (SNR) of the channel. In several cases the necessary conditions and sufficient conditions are shown to agree. In particular, we show that if the channel SNR is below a certain threshold, then an uncoded transmission scheme is optimal, and feedback is useless. We also show that, in general the source-channel separation approach is suboptimal, but that it is asymptotically optimal as the transmit power tends to infinity.

2 Problem Statement

2.1 Setup

Our setup is illustrated in Figure 1. A memoryless bivariate Gaussian source is con-

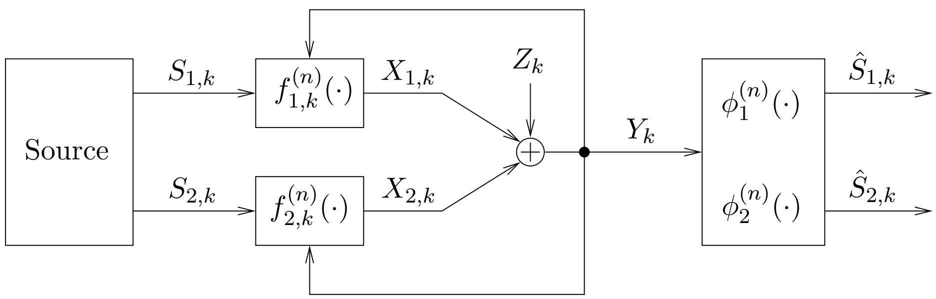

Figure 1: Bivariate Gaussian source with one-to-two Gaussian multiple-access channel with feedback.

nected to a two-to-one Gaussian multiple-access channel with perfect causal feedback. Each transmitter of the multiple-access channel observes one of the source components and wishes to describe it to the common receiver. The source symbols produced at time $k \in \mathbb{Z}$ are denoted by $(S_{1,k}, S_{2,k})$. The source output pairs $\{(S_{1,k}, S_{2,k})\}$ are independent identically distributed (IID) zero-mean Gaussians of covariance matrix

$$\mathsf{K}_{SS} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}, \tag{1}$$

where $\rho \in [-1, 1]$ and $0 < \sigma_i^2 < \infty$, $i \in \{1, 2\}$. The sequence $\{S_{1,k}\}$ of the first source component is observed by Transmitter 1 and the sequence $\{S_{2,k}\}$ of the second source component is observed by Transmitter 2. The two source components are to be described over the multiple-access channel to the common receiver by means of the channel input sequences $\{X_{1,k}\}$ and $\{X_{2,k}\}$, where $x_{1,k} \in \mathbb{R}$ and $x_{2,k} \in \mathbb{R}$. The corresponding time-k channel output is given by

$$Y_k = X_{1,k} + X_{2,k} + Z_k, \tag{2}$$

where Z_k is the time-k additive noise term, and where $\{Z_k\}$ are IID zero-mean variance-N Gaussian random variables that are independent of the source sequence.

We consider block encoding schemes and denote the block-length by n and the associated n-sequences in boldface, e.g. $\mathbf{S}_1 = (S_{1,1}, S_{1,2}, \ldots, S_{1,n})$. Transmitter $i \in \{1, 2\}$

is described by a sequence of functions $f_{i,k}^{(n)} \colon \mathbb{R}^n \times \mathbb{R}^{k-1} \to \mathbb{R}$, $k = 1, \ldots, n$, which, for every time instant $k \in \mathbb{R}$ produce the channel input $X_{i,k}$ from the source sequence $\mathbf{S}_i$ and the so-far-observed feedback sequence $Y^{k-1} = (Y_1, \ldots, Y_{k-1})$, i.e.

$$X_{i,k} = f_{i,k}^{(n)} \left(\mathbf{S}_i, Y^{k-1} \right) \qquad i \in \{1, 2\}. \tag{3}$$

The channel input sequences are subjected to expected average power constraints

$$\frac{1}{n} \sum_{k=1}^{n} \mathsf{E}\left[X_{i,k}^2\right] \leq P_i \qquad i \in \{1, 2\}, \tag{4}$$

for some given $P_i > 0$.

The receiver is described by two functions $\phi_i^{(n)} \colon \mathbb{R}^n \to \mathbb{R}^n$, $i \in \{1, 2\}$, each of which forms an estimate $\hat{\mathbf{S}}_i$ of the respective source sequence $\mathbf{S}_i$ based on the observed channel output sequence $\mathbf{Y}$. Thus,

$$\hat{\mathbf{S}}_i = \phi_i^{(n)} (\mathbf{Y}) \qquad i \in \{1, 2\}. \tag{5}$$

We are interested in the pairs of expected squared-error distortions that can be achieved simultaneously on the source-pair as the blocklength n tends to infinity. In view of this, we next define the notion of achievability.

2.2 Achievability of Distortion Pairs

Definition 2.1. Given $\sigma_1, \sigma_2 > 0$, $\rho \in [-1, 1]$, $P_1, P_2 > 0$, and $N > 0$ we say that the tuple $(D_1, D_2, \sigma_1^2, \sigma_2^2,\ \rho, P_1, P_2, N)$ is *achievable* if there exists a sequence of encoding functions $\left(\{f_{1,k}^{(n)}\}_{k=1}^n, \{f_{2,k}^{(n)}\}_{k=1}^n\right)$ as in (3), satisfying the average power constraints (4), and a sequence of reconstruction pairs $\left(\phi_1^{(n)}, \phi_2^{(n)}\right)$ as in (5), such that the average distortions resulting from these encoding and reconstruction functions fulfill

$$\varlimsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \mathsf{E}\left[\left(S_{i,k} - \hat{S}_{i,k} \right)^2 \right] \leq D_i, \quad i \in \{1, 2\},$$

whenever

$$Y_k = f_{1,k}^{(n)}(\mathbf{S}_1, Y^{k-1}) + f_{2,k}^{(n)}(\mathbf{S}_2, Y^{k-1}) + Z_k, \qquad \text{for } k \in \{1, 2, \ldots, n\},$$

and where $\{(S_{1,k}, S_{2,k})\}$ are IID zero-mean bivariate Gaussian vectors of covariance matrix K_{SS} as in (1) and $\{Z_k\}$ are IID zero-mean variance-N Gaussians that are independent of $\{(S_{1,k}, S_{2,k})\}$.

For given σ_1^2, σ_2^2, ρ, P_1, P_2, and N, we wish to find the set of pairs (D_1, D_2) such that $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2, N)$ is achievable. Sometimes, we will refer to the set of all (D_1, D_2) such that $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2, N)$ is achievable as the distortion region associated with $(\sigma_1^2, \sigma_2^2, \rho, P_1, P_2, N)$. In that sense, we will often say, with respect to some $(\sigma_1,\ \sigma_2,\ \rho,\ P_1,\ P_2, N)$, that the pair (D_1, D_2) is achievable, instead of saying that the tuple $(D_1, D_2, \sigma_1^2, \sigma_2^2,\ \rho, P_1, P_2, N)$ is achievable.

2.3 Normalization

For the described problem we now show that, without loss in generality, the source law given in (1) can be restricted to a simpler form. This restriction will ease the statement of our results as well as their derivations.

Reduction 2.1. For the problem stated in Sections 2.1 and 2.2, there is no loss in generality in restricting the source law to satisfy

$$\sigma_1^2 = \sigma_2^2 = \sigma^2 \qquad \text{and} \qquad \rho \in [0,1]. \tag{6}$$

Proof. The proof follows by noting that the described problem has certain symmetry properties with respect to the source law. We prove the reductions on the source variance and on the correlation coefficient separately.

i) The reduction to correlation coefficients $\rho \in [0,1]$ holds because the optimal distortion region depends on the correlation coefficient only via its absolute value $|\rho|$. That is, the tuple $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2, N)$ is achievable if, and only if, the tuple $(D_1, D_2, \sigma_1^2, \sigma_2^2, -\rho, P_1, P_2, N)$ is achievable. To see this, note that if $\left(\{f_{1,k}^{(n)}\}_{k=1}^n, \{f_{2,k}^{(n)}\}_{k=1}^n, \phi_1^{(n)}, \phi_2^{(n)}\right)$ achieves the distortion (D_1, D_2) for the source of correlation coefficient ρ, then $\left(\{\tilde{f}_{1,k}^{(n)}\}_{k=1}^n, \{f_{2,k}^{(n)}\}_{k=1}^n, \tilde{\phi}_1^{(n)}, \phi_2^{(n)}\right)$, where

$$\tilde{f}_{1,k}^{(n)}(\mathbf{S}_1, Y^{k-1}) = f_{1,k}^{(n)}(-\mathbf{S}_1, Y^{k-1}) \qquad \text{and} \qquad \tilde{\phi}_1^{(n)}(\mathbf{Y}) = -\phi_1^{(n)}(\mathbf{Y})$$

achieves (D_1, D_2) on the source with correlation coefficient $-\rho$.

ii) The restriction to source variances satisfying $\sigma_1^2 = \sigma_2^2 = \sigma^2$ incurs no loss of generality because the distortion region scales linearly with the source variances. That is, the tuple $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2, N)$ is achievable if, and only if, for every $\alpha_1, \alpha_2 \in \mathbb{R}^+$, the tuple $(\alpha_1 D_1, \alpha_2 D_2, \alpha_1 \sigma_1^2, \alpha_2 \sigma_2^2, \rho, P_1, P_2, N)$ is achievable. This can be seen as follows. If $\left(\{f_{1,k}^{(n)}\}_{k=1}^n, \{f_{2,k}^{(n)}\}_{k=1}^n, \phi_1^{(n)}, \phi_2^{(n)}\right)$ achieves the tuple $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2, N)$, then the combination of the encoders

$$\tilde{f}_{i,k}^{(n)}(\mathbf{S}_i, Y^{k-1}) = f_{i,k}^{(n)}(\mathbf{S}_i/\sqrt{\alpha_i}, Y^{k-1}), \qquad i \in \{1,2\},$$

with the reconstructors

$$\tilde{\phi}_i^{(n)}(\mathbf{Y}) = \sqrt{\alpha_i} \cdot \phi_i^{(n)}(\mathbf{Y}), \qquad i \in \{1,2\},$$

achieves the tuple $(\alpha_1 D_1, \alpha_2 D_2, \alpha_1 \sigma_1^2, \alpha_2 \sigma_2^2, \rho, P_1, P_2, N)$. And by an analogous argument it follows that if $(\alpha_1 D_1, \alpha_2 D_2, \alpha_1 \sigma_1^2, \alpha_2 \sigma_2^2, \rho, P_1, P_2, N)$ is achievable, then also $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2, N)$ is achievable. □

In view of Reduction 2.1 we assume for the remainder that the source law additionally satisfies (6).

2.4 "Symmetric Version" and a Convexity Property

The "symmetric version" of our problem corresponds to the case where the transmitters are subjected to the same power constraint, and where we seek to achieve the same distortion on each source component. That is, $P_1 = P_2 = P$, and we are interested in the minimal distortion

$$D^*(\sigma^2, \rho, P, N) \triangleq \inf\{D \colon (D, D, \sigma^2, \sigma^2, \rho, P, P, N) \text{ is achievable}\},$$

that is simultaneously achievable on $\{S_{1,k}\}$ and on $\{S_{2,k}\}$. In this case, we will often express the distortion $D^*(\sigma^2, \rho, P, N)$, for some fixed σ^2 and ρ, and as a function of the SNR P/N.

We conclude this section with a convexity property of the achievable distortions.

Remark 2.1. If $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2, N)$ and $(\tilde{D}_1, \tilde{D}_2, \sigma_1^2, \sigma_2^2, \rho, \tilde{P}_1, \tilde{P}_2, N)$ are achievable, then

$$\left(\lambda D_1 + \bar{\lambda}\tilde{D}_1, \lambda D_2 + \bar{\lambda}\tilde{D}_2, \sigma_1^2, \sigma_2^2, \rho, \lambda P_1 + \bar{\lambda}\tilde{P}_1, \lambda P_2 + \bar{\lambda}\tilde{P}_2, N\right),$$

is also achievable for every $\lambda \in [0,1]$, where $\bar{\lambda} = (1-\lambda)$.

Proof. Follows by a time-sharing argument. $\square$

3 Main Results

3.1 Necessary Condition for Achievability of (D_1, D_2)

To state our necessary condition we first introduce three rate-distortion functions. They are: the rate-distortion function $R_{S_1,S_2}(D_1, D_2)$ on $\{(S_{1,k}, S_{2,k})\}$; the rate-distortion function $R_{S_1|S_2}(D_1)$ on $\{S_{1,k}\}$, when the component $\{S_{2,k}\}$ is observed as side-information at both, encoder and decoder; and the rate-distortion function $R_{S_2|S_1}(D_2)$ on $\{S_{2,k}\}$ when the component $\{S_{1,k}\}$ is observed as side-information at both, encoder and decoder. For $\{(S_{1,k}, S_{2,k})\}$ jointly Gaussian as in (1) with $\sigma_1^2 = \sigma_2^2 = \sigma^2$, the two latter functions are given by

$$R_{S_1|S_2}(D_1) = \frac{1}{2}\log_2^+\left(\frac{\sigma^2(1-\rho^2)}{D_1}\right), \tag{7}$$

$$R_{S_2|S_1}(D_2) = \frac{1}{2}\log_2^+\left(\frac{\sigma^2(1-\rho^2)}{D_2}\right). \tag{8}$$

The function $R_{S_1,S_2}(D_1, D_2)$ is given in the following theorem.

Theorem 3.1 (Xiao, Luo [3]; Lapidoth, Tinguely [1, 2])**.** *The rate-distortion function $R_{S_1,S_2}(D_1, D_2)$ is given by*

$$R_{S_1,S_2}(D_1, D_2) = \begin{cases} \frac{1}{2}\log_2^+\left(\frac{\sigma^2}{D_{\min}}\right) & \text{if } (D_1, D_2) \in \mathscr{D}_1 \\ \frac{1}{2}\log_2^+\left(\frac{\sigma^4(1-\rho^2)}{D_1 D_2}\right) & \text{if } (D_1, D_2) \in \mathscr{D}_2 \\ \frac{1}{2}\log_2^+\left(\frac{\sigma^4(1-\rho^2)}{D_1 D_2 - \left(\rho\sigma^2 - \sqrt{(\sigma^2-D_1)(\sigma^2-D_2)}\right)^2}\right) & \text{if } (D_1, D_2) \in \mathscr{D}_3. \end{cases} \tag{9}$$

where $\log_2^+(x) = \max\{0, \log_2(x)\}$, $D_{\min} = \min\{D_1, D_2\}$ *and where the regions* $\mathscr{D}_1$, $\mathscr{D}_2$

and $\mathscr{D}_3$ *are given by*

$$\mathscr{D}_1 = \Bigg\{(D_1, D_2) :\ 0 \le D_1 \le \sigma^2(1-\rho^2),\ D_2 \ge \sigma^2(1-\rho^2) + \rho^2 D_1;$$
$$\sigma^2(1-\rho^2) < D_1 \le \sigma^2,\ D_2 \ge \sigma^2(1-\rho^2) + \rho^2 D_1,$$
$$D_2 \le \frac{D_1 - \sigma^2(1-\rho^2)}{\rho^2}\Bigg\},$$

$$\mathscr{D}_2 = \left\{(D_1, D_2) :\ 0 \le D_1 \le \sigma^2(1-\rho^2), 0 \le D_2 < (\sigma^2(1-\rho^2) - D_1)\frac{\sigma^2}{\sigma^2 - D_1}\right\},$$

$$\mathscr{D}_3 = \Bigg\{(D_1, D_2) :\ 0 \le D_1 \le \sigma^2(1-\rho^2),$$
$$(\sigma^2(1-\rho^2) - D_1)\frac{\sigma^2}{\sigma^2 - D_1} \le D_2 < \sigma^2(1-\rho^2) + \rho^2 D_1;$$
$$\sigma^2(1-\rho^2) < D_1 \le \sigma^2,\ \frac{D_1 - \sigma^2(1-\rho^2)}{\rho^2} < D_2 < \sigma^2(1-\rho^2) + \rho^2 D_1\Bigg\}.$$

Our necessary condition is now as follows.

Theorem 3.2. *A necessary condition for the achievability of* $(D_1, D_2, \sigma^2, \sigma^2, \rho, P_1, P_2, N)$ *is the existance of some* $\hat{\rho} \in [0, 1]$ *such that*

$$R_{S_1,S_2}(D_1, D_2) \le \frac{1}{2}\log_2\left(1 + \frac{P_1 + P_2 + 2\hat{\rho}\sqrt{P_1P_2}}{N}\right) \tag{10}$$

$$R_{S_1|S_2}(D_1) \le \frac{1}{2}\log_2\left(1 + \frac{P_1(1-\hat{\rho}^2)}{N}\right) \tag{11}$$

$$R_{S_2|S_1}(D_2) \le \frac{1}{2}\log_2\left(1 + \frac{P_2(1-\hat{\rho}^2)}{N}\right). \tag{12}$$

Proof. See Appendix A. □

We now specialize Theorem 3.2 to the symmetric case. To this end, we first substitute the rate-distortion functions $R_{S_1,S_2}(D_1, D_2)$, $R_{S_1|S_2}(D_1)$, $R_{S_2|S_1}(D_2)$ on the LHS of (10) – (12) by their explicit forms given in (9), (7), and (8) respectively. Substituting (D, D) for (D_1, D_2) in (10) & (9) yields that if (D, D) is achievable, then

$$D \ge \begin{cases} \frac{1}{2}\left(\frac{N\sigma^2(1+\rho)}{N+2P(1+\hat{\rho})} + \sigma^2(1-\rho)\right) & \text{if } \frac{P}{N} \le \frac{\rho}{1-\rho^2} \\ \sigma^2\sqrt{\frac{N(1-\rho^2)}{N+2P(1+\hat{\rho})}} & \text{if } \frac{P}{N} > \frac{\rho}{1-\rho^2}. \end{cases} \tag{13}$$

Similarly, from (11) & (7) [or (12) & (8)] we obtain that if (D, D) is achievable, then

$$D \ge \sigma^2 \frac{N(1-\rho^2)}{N + P(1-\hat{\rho}^2)}. \tag{14}$$

Denoting the RHS of (13) by $\xi(\sigma^2, \rho, P, N, \hat{\rho})$ and the RHS of (14) by $\psi(\sigma^2, \rho, P, N, \hat{\rho})$, gives the following lower bound on $D^*(\sigma^2, \rho, P, N)$:

Corollary 3.1. *In the symmetric case*

$$D^*(\sigma^2, \rho, P, N) \geq \min_{0\leq\hat{\rho}\leq 1} \max\left\{\xi(\sigma^2, \rho, P, N, \hat{\rho}), \psi(\sigma^2, \rho, P, N, \hat{\rho})\right\}.$$

The minimization over $\hat{\rho}$ is discussed in the following remark.

Remark 3.1. For $P/N \leq \rho^2/(2(1-\rho)(1+2\rho))$ the minimum in Corollary 3.1 is achieved by $\hat{\rho}^* = 1$, and for all larger P/N the minimum is achieved by the $\hat{\rho}^*$ satisfying

$$\xi(\sigma^2, \rho, P, N, \hat{\rho}^*) = \psi(\sigma^2, \rho, P, N, \hat{\rho}^*).$$

As $P/N \to \infty$ it can be shown that $\hat{\rho}^*$ tends to one and hence Corollary 3.1 yields

$$\liminf_{P/N\to\infty} \sqrt{\frac{P}{N}} D^*(\sigma^2, \rho, P, N) \geq \sigma^2 \sqrt{\frac{1-\rho^2}{4}}. \tag{15}$$

In the next section we show that the lim inf in (15) is a limit, and that it is achieved by source-channel separation.

3.2 Source-Channel Separation

We now consider the set of distortion pairs that are achieved by combining the optimal scheme for the corresponding source-coding problem with the optimal scheme for the corresponding channel-coding problem. The source-coding problem is illustrated in Figure 2. The two source components are observed by two separate encoders. These

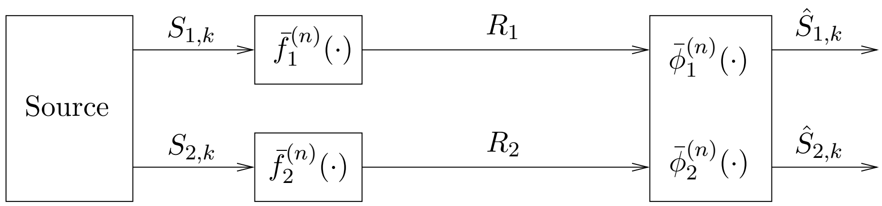

Figure 2: Distributed source coding problem for a bivariate Gaussian source.

two encoders wish to describe their source sequence to the common receiver by means of individual rate-limited and error-free bit pipes. The receiver estimates each of the sequences subject to expected squared-error distortion. A detailed description of this problem can be found in [4, 5]. The associated rate-distortion region is given in the next theorem.

Theorem 3.3 (Oohama [4]; Wagner, Tavildar, and Viswanath [5])**.** *For the Gaussian two-terminal source coding problem (with source components of unit variances) a distortion-pair* (D_1, D_2) *is achievable if, and only if,*

$$(R_1, R_2) \in \mathcal{R}_1(D_1) \cap \mathcal{R}_2(D_2) \cap \mathcal{R}_{\text{sum}}(D_1, D_2),$$

where

$$\mathcal{R}_1(D_1) = \left\{(R_1, R_2) : R_1 \geq \frac{1}{2}\log_2^+\left[\frac{1}{D_1}(1-\rho^2(1-2^{-2R_2}))\right]\right\}$$

$$\mathcal{R}_2(D_2) = \left\{(R_1, R_2) : R_2 \geq \frac{1}{2}\log_2^+\left[\frac{1}{D_2}(1-\rho^2(1-2^{-2R_1}))\right]\right\}$$

$$\mathcal{R}_{\text{sum}}(D_1, D_2) = \left\{(R_1, R_2) : R_1 + R_2 \geq \frac{1}{2}\log_2^+\left[\frac{(1-\rho^2)\beta(D_1, D_2)}{2D_1D_2}\right]\right\}$$

with

$$\beta(D_1, D_2) = 1 + \sqrt{1 + \frac{4\rho^2 D_1 D_2}{(1-\rho^2)^2}}.$$

The capacity region $\mathcal{C}_{FB}(P_1, P_2, N)$ of the Gaussian multiple-access channel with feedback was derived in [6] and is restated in the following theorem.

Theorem 3.4 (Ozarow [6])**.** *The capacity region* $\mathcal{C}_{FB}(P_1, P_2, N)$ *of the Gaussian multiple-access channel with perfect feedback is*

$$\begin{aligned}\mathcal{C}_{FB}(P_1, P_2, N) = \bigcup_{0 \le \bar{\rho} \le 1} \Bigg\{ (R_1, R_2) : R_1 &\le \frac{1}{2}\log_2\left(1 + \frac{P_1(1-\bar{\rho}^2)}{N}\right) \\ R_2 &\le \frac{1}{2}\log_2\left(1 + \frac{P_2(1-\bar{\rho}^2)}{N}\right) \\ R_1 + R_2 &\le \frac{1}{2}\log_2\left(1 + \frac{P_1 + P_2 + 2\bar{\rho}\sqrt{P_1 P_2}}{N}\right) \Bigg\}.\end{aligned}$$

The distortions achievable by source-channel separation are now given in the following Corollary.

Corollary 3.2. *A distortion pair* (D_1, D_2) *is achievable by source-channel separation if, and only if,*

$$\mathcal{R}(D_1, D_2) \cap \mathcal{C}_{FB}(P_1, P_2, N) \neq \emptyset.$$

From the sufficient condition of Corollary 3.2 and the necessary condition of Theorem 3.2 we can now derive the high-SNR asymptotics of an optimal scheme. To state these asymptotics, we denote by (D_1^*, D_2^*) an arbitrary distortion pair resulting from an optimal scheme.

Theorem 3.5 (High-SNR Distortion)**.** *The high-SNR asymptotic behavior of* (D_1^*, D_2^*) *is given by*

$$\lim_{N \to 0} \frac{P_1 + P_2 + 2\sqrt{P_1 P_2}}{N} D_1^* D_2^* = \sigma^4(1-\rho^2),$$

provided that $D_1^* \le \sigma^2$ *and* $D_2^* \le \sigma^2$*, and that*

$$\lim_{N \to 0} \frac{N}{P_1 D_1^*} = 0 \qquad \textit{and} \qquad \lim_{N \to 0} \frac{N}{P_2 D_2^*} = 0. \tag{16}$$

Proof. See Appendix B. □

Remark 3.2. The asymptotics of Theorem 3.5 are almost the same as those in [1, Theorem 4.5] for the setup without feedback. The only difference is that in the case with feedback the power term $P_1 + P_2 + 2\rho\sqrt{P_1 P_2}$ is replaced by $P_1 + P_2 + 2\sqrt{P_1 P_2}$. This stems from the fact that with feedback, as $P/N \to \infty$, the cooperation between the transmitters can be full.

Remark 3.3. Note that under source-channel separation, which achieves the high-SNR asymptotics, the cooperation between the transmitters takes place only at the channel-coding level. The source-coding is performed in a distributed manner.

To conclude this section we restate Theorem 3.5 more specifically for the symmetric case. Since there $D_1^* = D_2^* = D^*(\sigma^2, \rho, P, N)$, condition (16) is implicitly satisfied. Thus,

Corollary 3.3. *In the symmetric case*

$$\lim_{\frac{P}{N}\to\infty} \sqrt{\frac{P}{N}} D^*_{\mathrm{FB}}(\sigma^2, \rho, P, N) = \sigma^2 \sqrt{\frac{1-\rho^2}{4}}.$$

3.3 Uncoded Scheme

We now revisit the uncoded scheme of [1, Section 4.3], which was shown to be optimal for the setup without feedback whenever the SNR is below a certain threshold. For our setup with feedback, we show that this scheme is still optimal whenever the SNR is below the threshold of [1, Section 4.3]. This result implies that below this SNR-threshold feedback is useless.[1] Note, however, that feedback is beneficial for the source-channel separation approach because, even if noisy, it increases the capacity region of the Gaussian multiple-access channel [7].

The uncoded scheme operates as follows. Encoder $i \in \{1,2\}$ produces a time-k channel input $X_{i,k}$ which is a scaled version of the time-k source output $S_{i,k}$. The scaling is such that the average power constraint of the channel (4) is satisfied. That is,

$$X^{\mathrm{u}}_{i,k} = \sqrt{\frac{P_i}{\sigma^2}} S_{i,k} \qquad \text{for all } k \in \{1, 2, \ldots, n\}.$$

The decoder reconstructs the source output $S_{i,k}$ by performing the MMSE estimate of $S_{i,k}$, $i \in \{1,2\}$, $k \in \{1, 2, \ldots, n\}$, based on the time-$k$ channel output Y_k. That is,

$$\hat{S}^{\mathrm{u}}_{i,k} = \mathsf{E}[S_{i,k}|Y_k] \,.$$

The expected distortions $(D_1^{\mathrm{u}}, D_2^{\mathrm{u}})$ resulting from this uncoded scheme as well as its optimality below a certain SNR-threshold are stated in the following theorem.

Theorem 3.6. *The distortion pairs $(D_1^{\mathrm{u}}, D_2^{\mathrm{u}})$ resulting from the described uncoded scheme are given by*

$$D_1^{\mathrm{u}} = \sigma^2 \frac{(1-\rho^2)P_2 + N}{P_1 + P_2 + 2\rho\sqrt{P_1 P_2} + N} \qquad\qquad D_2^{\mathrm{u}} = \sigma^2 \frac{(1-\rho^2)P_1 + N}{P_1 + P_2 + 2\rho\sqrt{P_1 P_2} + N}.$$

These distortion pairs $(D_1^{\mathrm{u}}, D_2^{\mathrm{u}})$ are optimal, i.e., lie on the boundary of the distortion region, whenever

$$P_2(1-\rho^2)^2\Big(P_1 + 2\rho\sqrt{P_1 P_2}\Big) \le N\rho^2\Big(2P_2(1-\rho^2) + N\Big). \tag{17}$$

Proof. The expressions for D_1^{u} and D_2^{u} are derived in [1, Appendix D]. The optimality of the uncoded scheme is proven in Appendix C. For the particular case where P_1, P_2, N satisfy (17) with equality, the optimality can also be verified directly from Theorem 3.2. To this end, it suffices to notice that for $(D_1, D_2) = (D_1^{\mathrm{u}}, D_2^{\mathrm{u}})$, the necessary condition of Theorem 3.2 is satisfied with equality for $\hat{\rho}^* = \rho$. It thus follows that for any (D_1', D_2') satisfying $D_1' \le D_1^{\mathrm{u}}$ and $D_2' < D_2^{\mathrm{u}}$ or $D_1' < D_1^{\mathrm{u}}$ and $D_2' \le D_2^{\mathrm{u}}$ the necessary condition of Theorem 3.2 is violated for every $\hat{\rho} \in [-1, 1]$. And hence, the uncoded scheme is optimal. ☐

[1] By the simple structure of the uncoded scheme, it follows that feedback is useless not only in terms of performance, but also in terms of delay and complexity.

Corollary 3.4. *Source-channel separation is in general suboptimal.*

Proof. This can be verified by comparing the achievable distortions given in Corollary 3.2 with the achievable distortions given in Theorem 3.6. For example, in the symmetric case it can be verified that for all $\rho > 0$ and $P/N \leq \rho/(1-\rho^2)$, the smallest distortions achievable by source-channel separation (Corollary 3.2) are strictly larger than the distortions resulting from the optimal uncoded scheme (Theorem 3.6). $\square$

Remark 3.4. From Theorem 3.6 it follows that if P_1, P_2, N satisfy (17) with a strict inequality, then the necessary condition of Theorem 3.2 is not sufficient. This is due to the constraints (11) and (12) which are loose at low SNRs and is best seen in the symmetric case. In the symmetric case, Theorem 3.2 (cf. (11) and (12)) yields that for (D, D) to be achievable, it is necessary that D satisfy

$$D \geq \sigma^2(1-\rho^2)\frac{N}{N + P(1-\hat{\rho}^2)}, \tag{18}$$

i.e., that (14) hold. Since $\hat{\rho} \in [0, 1]$, the RHS of (18) is upper bounded by $\sigma^2(1-\rho^2)$. Thus, for sufficiently low SNRs the constraint of (18) is inactive, and the only active constraint is the one of (13). But, if only (13) is active, then $\hat{\rho}^* = 1$, which corresponds to fully cooperating transmitters, and thus, yields a loose lower bound on $D^*(\sigma^2, \rho, P, N)$ at low SNRs.

We conclude the section on our main results by restating Theorem 3.6 more specifically for the symmetric case.

Corollary 3.5. *In the symmetric case*

$$D^*(\sigma^2, \rho, P, N) = \sigma^2 \frac{P(1-\rho^2) + N}{2P(1+\rho) + N}, \qquad \frac{P}{N} \leq \frac{\rho}{1-\rho^2}. \tag{19}$$

4 Summary

We studied the power-versus-distortion trade-off for the transmission of a memoryless bivariate Gaussian source over a two-to-one average-power limited Gaussian multiple-access channel with perfect causal feedback. In this problem, each of two separate transmitters observes a different component of a memoryless bivariate Gaussian source as well as the feedback from the channel output of the previous time-instants. Based on the observed source sequence and the feedback, each transmitter then describes its source component to the common receiver via an average-power constrained Gaussian multiple-access channel. From the resulting channel output, the receiver wishes to reconstruct both source components with the least possible expected squared-error distortion. Our interest was in the set of distortion pairs that can be achieved by the receiver on the two source components. Our main results were:

- A necessary condition (Theorem 3.2) for the achievability of a distortion pair (D_1, D_2).

- The precise high-SNR asymptotic behaviour (Theorem 3.5) of optimal transmission schemes, which in the symmetric case (Corollary 3.3) is given by

$$\lim_{P/N\to\infty} \sqrt{\frac{P}{N}} D^*(\sigma^2, \rho, P, N) = \sigma^2 \sqrt{\frac{1-\rho^2}{4}},$$

and which is shown to be achievable by source-channel separation.

- The optimality, for all SNRs below a certain threshold, of an uncoded transmission scheme, which ignores the feedback (Theorem 3.6). In the symmetric case, this optimality result (Corollary 3.5) is given by

$$D^*(\sigma^2, \rho, P, N) = \sigma^2 \frac{P(1-\rho^2)+N}{2P(1+\rho)+N}, \qquad \frac{P}{N} \le \frac{\rho}{1-\rho^2}.$$

A Proof of Theorem 3.2

In Theorem 3.2 we have given a necessary condition for the achievability of a distortion pair (D_1, D_2) for the multiple-access problem with feedback. The proof of this necessary condition uses the following two lemmas.

Lemma A.1. *For our multiple-access setup with feedback, let $\{X_{1,k}\}$, $\{X_{2,k}\}$ and $\{Y_k\}$ be the channel inputs and channel outputs of a coding scheme achieving some distortion pair (D_1, D_2). Then, for every $\delta > 0$ there exists an $n_0(\delta) > 0$ such that for all $n > n_0(\delta)$*

$$nR_{S_1,S_2}(D_1+\delta, D_2+\delta) \le \sum_{k=1}^{n} I(X_{1,k}, X_{2,k}; Y_k), \tag{20}$$

$$nR_{S_1|S_2}(D_1+\delta) \le \sum_{k=1}^{n} I(X_{1,k}; Y_k|X_{2,k}), \tag{21}$$

$$nR_{S_2|S_1}(D_2+\delta) \le \sum_{k=1}^{n} I(X_{2,k}; Y_k|X_{1,k}). \tag{22}$$

Proof. The proofs of (20) – (22) follow along the lines of the proof for the univariate analog (see e.g. [8, page 15]). The main ingredients in those derivations are the convexity of the rate-distortion functions and the data-processing inequality. We start with the proof of (20). By the definition of an achievable distortion pair (D_1, D_2) (Definition 2.1) and by the monotonicity of $R_{S_1,S_2}(\Delta_1, \Delta_2)$ in (Δ_1, Δ_2), we have that for every $\delta > 0$ there exists an $n_0(\delta) > 0$ such that for every $n > n_0(\delta)$

$$\begin{aligned}
nR_{S_1,S_2}(D_1+\delta, D_2+\delta) &\le nR_{S_1,S_2}\left(\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[(S_{1,k}-\hat{S}_{1,k})^2\right], \frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[(S_{2,k}-\hat{S}_{2,k})^2\right]\right) \\
&\overset{a)}{\le} n\sum_{k=1}^{n} \frac{1}{n} R_{S_1,S_2}\Bigg(\underbrace{\mathsf{E}\left[(S_{1,k}-\hat{S}_{1,k})^2\right]}_{d_{1,k}}, \underbrace{\mathsf{E}\left[(S_{2,k}-\hat{S}_{2,k})^2\right]}_{d_{2,k}}\Bigg) \\
&= \sum_{k=1}^{n} \min_{\substack{P_{T_1,T_2|S_1,S_2}: \\ \mathsf{E}\left[(S_1-T_1)^2\right] \le d_{1,k} \\ \mathsf{E}\left[(S_2-T_2)^2\right] \le d_{2,k}}} I(S_1, S_2; T_1, T_2) \\
&\le \sum_{k=1}^{n} I(S_{1,k}, S_{2,k}; \hat{S}_{1,k}, \hat{S}_{2,k}) \\
&= \sum_{k=1}^{n} h(S_{1,k}, S_{2,k}) - \sum_{k=1}^{n} h(S_{1,k}, S_{2,k}|\hat{S}_{1,k}, \hat{S}_{2,k})
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=1}^{n} h(S_{1,k}, S_{2,k}) - \sum_{k=1}^{n} h(S_{1,k}, S_{2,k} | \hat{\mathbf{S}}_1, \hat{\mathbf{S}}_2, S_{1,1}^{k-1}, S_{2,1}^{k-1}) \\
&= h(\mathbf{S}_1, \mathbf{S}_2) - h(\mathbf{S}_1, \mathbf{S}_2 | \hat{\mathbf{S}}_1, \hat{\mathbf{S}}_2) \\
&= I(\mathbf{S}_1, \mathbf{S}_2; \hat{\mathbf{S}}_1, \hat{\mathbf{S}}_2) \\
&\overset{b)}{\leq} I(\mathbf{S}_1, \mathbf{S}_2; \mathbf{Y}),
\end{aligned} \tag{23}
$$

where in step $a)$ we have used of the convexity of $R_{S_1,S_2}(D_1, D_2)$, and in step $b)$ we have used the data-processing inequality. The RHS of (23) can be further bounded as follows

$$
\begin{aligned}
I(\mathbf{S}_1, \mathbf{S}_2; \mathbf{Y}) &= h(\mathbf{Y}) - h(\mathbf{Y} | \mathbf{S}_1, \mathbf{S}_2) \\
&= h(\mathbf{Y}) - \sum_{k=1}^{n} h(Y_k | \mathbf{S}_1, \mathbf{S}_2, Y_1^{k-1}) \\
&\leq h(\mathbf{Y}) - \sum_{k=1}^{n} h(Y_k | \mathbf{S}_1, \mathbf{S}_2, Y_1^{k-1}, X_{1,k}, X_{2,k}) \\
&\overset{a)}{=} h(\mathbf{Y}) - \sum_{k=1}^{n} h(Y_k | X_{1,k}, X_{2,k}) \\
&\leq \sum_{k=1}^{n} h(Y_k) - \sum_{k=1}^{n} h(Y_k | X_{1,k}, X_{2,k}) \\
&= \sum_{k=1}^{n} I(X_{1,k}, X_{2,k}; Y_k),
\end{aligned} \tag{24}
$$

where inequality $a)$ follows because given the channel inputs $X_{1,k}$, $X_{2,k}$, the channel output Y_k is independent of $(\mathbf{S}_1, \mathbf{S}_2, Y_1^{k-1})$. Inequalities (23) and (24) combine to prove (20).

The derivations for (21) and (22) are similar to the one for (20). Since there is a symmetry between the derivation of (21) and the derivation of (22), we only give the derivation of (21). By the definition of an achievable distortion pair (D_1, D_2) and by the monotonicity of $R_{S_1|S_2}(\Delta_1)$ in Δ_1, we have that for every $\delta > 0$ there exists an $n_0(\delta) > 0$ such that for every $n > n_0(\delta)$ we have

$$
\begin{aligned}
nR_{S_1|S_2}(D_1 + \delta) &\leq nR_{S_1|S_2}\left(\frac{1}{n} \sum_{k=1}^{n} \mathsf{E}\left[(S_{1,k} - \hat{S}_{1,k})^2 \right] \right) \\
&\overset{a)}{\leq} n \sum_{k=1}^{n} \frac{1}{n} R_{S_1|S_2}\Big(\underbrace{\mathsf{E}\left[(S_{1,k} - \hat{S}_{1,k})^2 \right]}_{d_{1,k}} \Big) \\
&= \sum_{k=1}^{n} \min_{\substack{P_{T_k|S_{1,k},S_{2,k}}: \\ \mathsf{E}\left[(S_{1,k} - T_k)^2\right] \leq d_{1,k}}} I(S_{1,k}; T_k | S_{2,k}) \\
&\leq \sum_{k=1}^{n} I(S_{1,k}; \hat{S}_{1,k} | S_{2,k}) \\
&= \sum_{k=1}^{n} h(S_{1,k} | S_{2,k}) - \sum_{k=1}^{n} h(S_{1,k} | \hat{S}_{1,k}, S_{2,k})
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} h(S_{1,k}|S_{1,1}^{k-1}, \mathbf{S}_2) - \sum_{k=1}^{n} h(S_{1,k}|\hat{S}_{1,k}, S_{2,k}) \\
&\leq \sum_{k=1}^{n} h(S_{1,k}|S_{1,1}^{k-1}, \mathbf{S}_2) - \sum_{k=1}^{n} h(S_{1,k}|\hat{\mathbf{S}}_1, \mathbf{S}_2, S_{1,1}^{k-1}) \\
&= \sum_{k=1}^{n} I(S_{1,k}; \hat{\mathbf{S}}_1|\mathbf{S}_2, S_{1,1}^{k-1}) \\
&= I(\mathbf{S}_1; \hat{\mathbf{S}}_1|\mathbf{S}_2) \\
&\overset{b)}{\leq} I(\mathbf{S}_1, \mathbf{Y}|\mathbf{S}_2),
\end{aligned} \tag{25}
$$

where step $a)$ follows by the convexity of $R_{S_1|S_2}(D_1)$ and step $b)$ follows by the data-processing in equality, i.e.

$$
\begin{aligned}
I(\mathbf{S}_1; \mathbf{Y}, \hat{\mathbf{S}}_1|\mathbf{S}_2) &= I(\mathbf{S}_1; \hat{\mathbf{S}}_1|\mathbf{S}_2) - \underbrace{I(\mathbf{S}_1; \mathbf{Y}|\hat{\mathbf{S}}_1, \mathbf{S}_2)}_{\geq 0} \\
&= I(\mathbf{S}_1; \mathbf{Y}|\mathbf{S}_2) - \underbrace{I(\mathbf{S}_1; \hat{\mathbf{S}}_1|\mathbf{Y}, \mathbf{S}_2)}_{=0}.
\end{aligned}
$$

The RHS of (25) can be further bounded as follows

$$
\begin{aligned}
I(\mathbf{S}_1; \mathbf{Y}|\mathbf{S}_2) &= h(\mathbf{Y}|\mathbf{S}_2) - h(\mathbf{Y}|\mathbf{S}_1, \mathbf{S}_2) \\
&= \sum_{k=1}^{n} h(Y_k|Y_1^{k-1}, \mathbf{S}_2) - \sum_{k=1}^{n} h(Y_k|\mathbf{S}_1, \mathbf{S}_2, Y_1^{k-1}) \\
&= \sum_{k=1}^{n} h(Y_k|Y_1^{k-1}, \mathbf{S}_2, X_{2,k}) - \sum_{k=1}^{n} h(Y_k|\mathbf{S}_1, \mathbf{S}_2, Y_1^{k-1}, X_{1,k}, X_{2,k}) \\
&\overset{a)}{\leq} \sum_{k=1}^{n} h(Y_k|X_{2,k}) - \sum_{k=1}^{n} h(Y_k|X_{1,k}, X_{2,k}) \\
&= \sum_{k=1}^{n} I(X_{1,k}; Y_k|X_{2,k}),
\end{aligned} \tag{26}
$$

where $a)$ follows because given the channel inputs $X_{1,k}$, $X_{2,k}$, the channel output Y_k is independent of $(\mathbf{S}_1, \mathbf{S}_2, Y_1^{k-1})$. Inequalities (25) and (26) combine to prove (21). □

Lemma A.2. *Let $\{X_{1,k}\}$ and $\{X_{2,k}\}$ be zero-mean sequences satisfying $\sum_{i=1}^{n} \mathsf{E}\left[X_{i,k}^2\right] \leq nP_i$, $i \in \{1, 2\}$. Let $Y_k = X_{1,k} + X_{2,k} + Z_k$, where $\{Z_k\}$ are IID zero-mean variance-N Gaussian, and where for every k, Z_k is independent of $(X_{1,k}, X_{2,k})$. Let $\hat{\rho}_n \in [0, 1]$ be given by*

$$
\hat{\rho}_n \triangleq \frac{\left|\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}[X_{1,k}X_{2,k}]\right|}{\sqrt{\left(\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[X_{1,k}^2\right]\right)\left(\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[X_{2,k}^2\right]\right)}}. \tag{27}
$$

Then

$$
\sum_{k=1}^{n} I(X_{1,k}, X_{2,k}; Y_k) \leq \frac{n}{2}\log_2\left(1 + \frac{P_1 + P_2 + 2\hat{\rho}_n\sqrt{P_1P_2}}{N}\right), \tag{28}
$$

$$
\sum_{k=1}^{n} I(X_{1,k}; Y_k|X_{2,k}) \leq \frac{n}{2}\log_2\left(1 + \frac{P_1(1 - \hat{\rho}_n^2)}{N}\right), \tag{29}
$$

$$\sum_{k=1}^{n} I(X_{2,k};Y_k|X_{1,k}) \leq \frac{n}{2}\log_2\left(1+\frac{P_2(1-\hat{\rho}_n^2)}{N}\right). \tag{30}$$

Proof. See [6, pp. 627]. □

Proof of Theorem 3.2. The proof now follows by jointly bounding the expressions on the RHS of (20), (21), and (22) by means of Lemma A.2, and using that for $n \to \infty$ Lemma A.1 holds for every $\delta > 0$. □

B Proof of Theorem 3.5

For $\rho = 1$ the result follows by noting that the multiple-access problem reduces to a point-to-point problem where $D_1^* = D_2^*$. Hence, we shall now assume

$$\rho < 1. \tag{31}$$

The result can then be obtained from the necessary condition for the achievability of a distortion pair (D_1, D_2) in Theorem 3.2 and from the sufficient conditions for the achievability of a distortion pair (D_1, D_2) that follow from source-channel separation in Corollary 3.2.

By Corollary 3.2 it follows that a distortion pair $(\bar{D}_1, \bar{D}_2)$ is achievable if $\bar{D}_1 \leq \sigma^2$, $\bar{D}_2 \leq \sigma^2$ and

$$\bar{D}_1 \geq \sigma^2 2^{-2R_1}(1-\rho^2) + \sigma^2\rho^2 2^{-2(R_1+R_2)} \tag{32}$$

$$\bar{D}_2 \geq \sigma^2 2^{-2R_2}(1-\rho^2) + \sigma^2\rho^2 2^{-2(R_1+R_2)} \tag{33}$$

$$\bar{D}_1\bar{D}_2 = \sigma^4 2^{-2(R_1+R_2)}(1-\rho^2) + \sigma^4\rho^2 2^{-4(R_1+R_2)}, \tag{34}$$

where the rate-pair (R_1, R_2) satisfies for some $\bar{\rho} \in [0,1]$

$$R_1 \leq \frac{1}{2}\log_2\left(\frac{P_1(1-\bar{\rho}^2)}{N}\right) \tag{35}$$

$$R_2 \leq \frac{1}{2}\log_2\left(\frac{P_2(1-\bar{\rho}^2)}{N}\right) \tag{36}$$

$$R_1 + R_2 \leq \frac{1}{2}\log_2\left(\frac{P_1+P_2+2\bar{\rho}\sqrt{P_1P_2}}{N}\right). \tag{37}$$

If we restrict ourselves to distortion pairs $(\bar{D}_1, \bar{D}_2)$ satisfying

$$\lim_{N\to 0}\frac{N}{P_1\bar{D}_1} = 0 \qquad \text{and} \qquad \lim_{N\to 0}\frac{N}{P_2\bar{D}_2} = 0, \tag{38}$$

and to ρ satisfying (31), then for sufficiently small $N > 0$ the constraints (32) and (33) become redundant. Consequently, for N sufficiently small, any distortion pair $(\bar{D}_1, \bar{D}_2)$ satisfying (38) and (34), where (R_1, R_2) satisfies (35)–(37) for some $\bar{\rho} \in [0,1]$, is achievable. And because for any fixed $\bar{\rho} \in [0,1)$ as $N \to 0$ the Constraints (35) and (36) become redundant, it follows that any distortion pair satisfying (38) and

$$\lim_{N\to 0}\frac{P_1+P_2+2\bar{\rho}\sqrt{P_1P_2}}{N}\bar{D}_1\bar{D}_2 = \sigma^4(1-\rho^2), \tag{39}$$

for some $\bar{\rho} \in [0, 1)$, is achievable. Since $\bar{\rho}$ can be chosen arbitrarily close to 1, a simple calculus argument shows that

$$\lim_{N \to 0} \frac{P_1 + P_2 + 2\sqrt{P_1 P_2}}{N} \bar{D}_1 \bar{D}_2 = \sigma^4 (1 - \rho^2), \tag{40}$$

is achievable.

Next, let $\big(D_1^*(\sigma^2, \rho, P_1, P_2, N), D_2^*(\sigma^2, \rho, P_1, P_2, N)\big)$ be a distortion pair resulting from an arbitrary optimal scheme for the corresponding SNR, and let (D_1^*, D_2^*) be the corresponding shorthand notation for this distortion pair. By Theorem 3.2 we have that

$$R_{S_1,S_2}(D_1, D_2) \le \frac{1}{2} \log_2 \left(1 + \frac{P_1 + P_2 + 2\sqrt{P_1 P_2}}{N} \right). \tag{41}$$

If (D_1^*, D_2^*) satisfies

$$\lim_{N \to 0} \frac{N}{P_1 D_1^*} = 0 \qquad \text{and} \qquad \lim_{N \to 0} \frac{N}{P_2 D_2^*} = 0, \tag{42}$$

then for N sufficiently small

$$R_{S_1,S_2}(D_1^*, D_2^*) = \frac{1}{2} \log_2^+ \left(\frac{\sigma^4 (1 - \rho^2)}{D_1^* D_2^*} \right), \tag{43}$$

by Theorem 3.1 and because $(D_1^*, D_2^*) \in \mathscr{D}_2$. From (41) and (43) we thus get that if (D_1^*, D_2^*) satisfies (42), then

$$\lim_{N \to 0} \frac{P_1 + P_2 + 2\sqrt{P_1 P_2}}{N} D_1^* D_2^* \ge \sigma^4 (1 - \rho^2). \tag{44}$$

Combining (40) with (44) yields Theorem 3.5. □

C Proof of Theorem 3.6

Theorem 3.6 states that for the multiple-access problem with feedback, if P_1, P_2, N satisfy (17), then the uncoded scheme is optimal, i.e. no pair (D_1, D_2) satisfying $D_1 \le D_1^{\mathrm{u}}$ and $D_2 < D_2^{\mathrm{u}}$ or satisfying $D_1 < D_1^{\mathrm{u}}$ and $D_2 \le D_2^{\mathrm{u}}$ is achievable. For P_1, P_2, N satisfying (17) with equality this was proven right after Theorem 3.6. Thus, here we restrict ourselves to P_1, P_2, N satisfying (17) with strict inequality.

We now show the inachievability of every (D_1, D_2) satisfying $D_1 < D_1^{\mathrm{u}}$ and $D_2 \le D_2^{\mathrm{u}}$. The inachievability of every (D_1, D_2) satisfying $D_1 \le D_1^{\mathrm{u}}$ and $D_2 < D_2^{\mathrm{u}}$ follows by similar arguments and is therefore omitted. The main step in our proof follows by contradiction. More precisely, we show that a contradiction arises from the following assumption.

Assumption C.1 (Leading to a contradiction)**.** For P_1, P_2, N satisfying (17) with strict inequality, there exist encoding rules $\{f_{i,k}^{(n)}\}$ satisfying the average power constraints (4), which, when combined with the optimal conditional expectation reconstructors

$$\hat{\mathbf{S}}_i = \mathsf{E}[\mathbf{S}_i | \mathbf{Y}], \qquad i \in \{1, 2\}, \tag{45}$$

result in

$$\varlimsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[(S_{i,k}-\hat{S}_{i,k})^2\right] \triangleq D_i^* \qquad\qquad i\in\{1,2\}, \tag{46}$$

such that

$$(D_1^*, D_2^*) \in \operatorname{int}(\mathscr{D}_3), \qquad D_1^* < D_1^{\mathrm{u}} \qquad \text{and} \qquad D_2^* = D_2^{\mathrm{u}}, \tag{47}$$

where we have denoted by $\operatorname{int}(\mathscr{D}_3)$ the interior of $\mathscr{D}_3$.

Once a contradiction from Assumption C.1 is established, it will follow that Assumption C.1 is false and the proof of Theorem 3.6 will follow in Section C.3.

Assume that Assumption C.1 is true. Let $\{f_{i,k}^{(n)}\}$ be a sequence of encoding functions, with resulting channel inputs $\{X_{1,k}, X_{2,k}\}$ and resulting channel outputs $\{Y_k\}$, which, when combined with the optimal conditional expectation reconstructors $\hat{\mathbf{S}}_1 = \mathsf{E}\left[\mathbf{S}_1\middle|\mathbf{Y}\right]$ and $\hat{\mathbf{S}}_2 = \mathsf{E}\left[\mathbf{S}_2\middle|\mathbf{Y}\right]$ result in distortions (D_1^*, D_2^*) as defined in (46) and satisfying (47). The contradiction based on Assumption C.1 will be obtained by deriving contradictory lower and upper bounds for the expected squared-error that Transmitter 2 can achieve at the end of the transmission on the sequence $\mathbf{W} \triangleq \mathbf{S}_1 - \rho\mathbf{S}_2$. To this end, let $\varphi^{(n)}(\mathbf{S}_2, \mathbf{Y})$ be some estimator of $\mathbf{W}$ from $(\mathbf{S}_2, \mathbf{Y})$ and let $D_W(\varphi^{(n)})$ be the mean squared-error associated with it:

$$D_W(\varphi^{(n)}) \triangleq \frac{1}{n}\mathsf{E}\left[\|\mathbf{W} - \varphi^{(n)}(\mathbf{S}_2, \mathbf{Y})\|^2\right].$$

Based on Assumption C.1, we now derive a lower bound on $D_W(\varphi^{(n)})$.

C.1 "Lower Bound" on $D_W(\varphi^{(n)})$

In this section we show that

$$\text{Assumption C.1} \quad \Rightarrow \quad \left(\varliminf_{n\to\infty} D_W(\varphi^{(n)}) > \sigma^2(1-\rho^2)\frac{N}{N+P_1(1-\rho^2)} \qquad \forall \varphi^{(n)}\right). \tag{48}$$

The idea in showing (48) is to exploit the fact that the sequence $\mathbf{W}$ is independent of $\mathbf{S}_2$, and that therefore the only information that Transmitter 2 receives about $\mathbf{W}$ is via the feedback signal $\mathbf{Y}$. Roughly speaking, we then show that if $\mathbf{Y}$ allows for "good" estimates of $\mathbf{S}_1$ and $\mathbf{S}_2$, i.e. if $D_1^* < D_1^{\mathrm{u}}$ and $D_2^* = D_2^{\mathrm{u}}$, then $\mathbf{Y}$ can only contain "little" information about $\mathbf{W}$, and hence Transmitter 2 can only make a coarse estimate of $\mathbf{W}$. The main element in showing (48) is given by the following lemma.

Lemma C.1. *Let $\hat{\rho}_n$ be as defined in* (27)*. Then*

$$I(\mathbf{S}_1; \mathbf{Y}|\mathbf{S}_2) \le \frac{n}{2}\log_2\left(1 + \frac{P_1(1-\hat{\rho}_n^2)}{N}\right)$$

and

$$\text{Assumption C.1} \qquad \Rightarrow \qquad \varliminf_{n\to\infty} \hat{\rho}_n > \rho.$$

Proof. Combining (26) with Lemma A.2 Inequality (29) gives

$$I(\mathbf{S}_1; \mathbf{Y}|\mathbf{S}_2) \le \frac{n}{2}\log_2\left(1 + \frac{P_1(1-\hat{\rho}_n^2)}{N}\right),$$

with $\hat{\rho}_n$ as defined in (27). It now remains to show that Assumption C.1 implies that $\varliminf_{n\to\infty} \hat{\rho}_n > \rho$. To this end, we recall that from [1, Proof of Theorem 4.1] we have that if P_1, P_2, N satisfy (17), then the corresponding $(D_1^{\mathrm{u}}, D_2^{\mathrm{u}})$ satisfies [1, Condition (14) of Theorem 4.1] with equality, i.e.,

$$R_{S_1,S_2}(D_1^{\mathrm{u}}, D_2^{\mathrm{u}}) = \frac{1}{2}\log_2\left(1 + \frac{P_1 + P_2 + 2\rho\sqrt{P_1P_2}}{N}\right). \tag{49}$$

Next, we notice that since Assumption C.1 guarantees that (D_1^*, D_2^*) is achievable, it follows from Lemma A.1 that for every $\delta > 0$ there exists an $n'(\delta) > 0$ such that for all $n > n'(\delta)$ we have

$$\begin{aligned} nR_{S_1,S_2}(D_1^* + \delta, D_2^* + \delta) &\overset{a)}{\leq} \sum_{k=1}^{n} I(X_{1,k}, X_{2,k}; Y_k) \\ &\overset{b)}{\leq} \frac{n}{2}\log_2\left(1 + \frac{P_1 + P_2 + 2\hat{\rho}_n\sqrt{P_1P_2}}{N}\right), \end{aligned} \tag{50}$$

where $a)$ follows from (20) in Lemma A.1, and $b)$ follows from Lemma A.2. Taking the $\liminf$ of (50) yields that for every $\delta > 0$

$$R_{S_1,S_2}(D_1^* + \delta, D_2^* + \delta) \leq \frac{1}{2}\log_2\left(1 + \frac{P_1 + P_2 + 2\hat{\rho}^*\sqrt{P_1P_2}}{N}\right),$$

where $\hat{\rho}^* = \varliminf_{n\to\infty} \hat{\rho}_n$. And since $R_{S_1,S_2}(D_1, D_2)$ is continuous in (D_1, D_2) it follows, upon letting δ tend to zero, that

$$R_{S_1,S_2}(D_1^*, D_2^*) \leq \frac{1}{2}\log_2\left(1 + \frac{P_1 + P_2 + 2\hat{\rho}^*\sqrt{P_1P_2}}{N}\right). \tag{51}$$

By Assumption C.1 and by the strict monotonicity of $R_{S_1,S_2}(D_1, D_2)$ as a function of D_1 in $\mathrm{int}(\mathscr{D}_3)$, it follows from the hypothesis $D_1^* < D_1^{\mathrm{u}}$ and $D_1^* = D_1^{\mathrm{u}}$ that

$$R_{S_1,S_2}(D_1^{\mathrm{u}}, D_2^{\mathrm{u}}) < R_{S_1,S_2}(D_1^*, D_2^*). \tag{52}$$

Combining (52) with (51) and (49) gives $\varliminf_{n\to\infty} \hat{\rho}_n > \rho$. $\square$

We next prove that

$$D_W(\varphi^{(n)}) \geq \sigma^2(1 - \rho^2)2^{-\frac{2}{n}I(\mathbf{S}_1;\mathbf{Y}|\mathbf{S}_2)}. \tag{53}$$

To derive (53), denote by $R_W(D)$ the rate-distortion function for a source of the law of $\mathbf{W}$. We then have

$$\begin{aligned} nR_W(D_W(\varphi^{(n)})) &\overset{a)}{\leq} I(\mathbf{W}; \varphi^{(n)}(\mathbf{S}_2, \mathbf{Y})) \\ &\overset{b)}{\leq} I(\mathbf{W}; \mathbf{Y}, \mathbf{S}_2) \\ &= I(\mathbf{S}_1 - \rho\mathbf{S}_2; \mathbf{Y}, \mathbf{S}_2) \\ &= h(\mathbf{S}_1 - \rho\mathbf{S}_2) - h(\mathbf{S}_1 - \rho\mathbf{S}_2|\mathbf{Y}, \mathbf{S}_2) \\ &\overset{c)}{=} h(\mathbf{S}_1 - \rho\mathbf{S}_2|\mathbf{S}_2) - h(\mathbf{S}_1 - \rho\mathbf{S}_2|\mathbf{Y}, \mathbf{S}_2) \\ &= h(\mathbf{S}_1|\mathbf{S}_2) - h(\mathbf{S}_1|\mathbf{Y}, \mathbf{S}_2) \\ &= I(\mathbf{S}_1; \mathbf{Y}|\mathbf{S}_2), \end{aligned} \tag{54}$$

where inequality a) follows by the data-processing inequality and the convexity of $R_W(\cdot)$. Inequality b) follows by the data-processing inequality, and c) follows since $\mathbf{S}_2$ and $\mathbf{S}_1 - \rho\mathbf{S}_2$ are independent. Substituting $R_W(D_W(\varphi^{(n)}))$ on the LHS of (54) by its explicit form gives

$$\frac{n}{2}\log_2\left(\frac{\sigma^2(1-\rho^2)}{D_W(\varphi^{(n)})}\right) \leq I(\mathbf{S}_1;\mathbf{Y}|\mathbf{S}_2).$$

Rewriting this inequality establishes (53).

Lemma C.1 and Inequality (53) combine to prove (48). We next derive an upper bound on $D_W(\varphi^{(n)})$.

C.2 "Upper Bound" on minimal $D_W(\varphi^{(n)})$

We now present an estimator $\tilde{\varphi}^{(n)}(\mathbf{S}_2,\mathbf{Y})$ for which we show that

$$\text{Assumption C.1} \quad \Rightarrow \quad \varlimsup_{\nu\to\infty} D_W(\tilde{\varphi}^{(n_\nu)}) < \sigma^2(1-\rho^2)\frac{N}{N+P_1(1-\rho^2)}, \tag{55}$$

for some monotonically increasing sequence $\{n_\nu\}$ of integers. From Implications (55) and (48) we then conclude that Assumption C.1 is false. The estimator $\tilde{\varphi}^{(n)}(\mathbf{S}_2,\mathbf{Y})$ is given by

$$\begin{aligned}\tilde{\varphi}^{(n)}(\mathbf{S}_2,\mathbf{Y}) &\triangleq \alpha\hat{\mathbf{S}}_1 - \beta\mathbf{S}_2\\ &= \alpha\mathsf{E}[\mathbf{S}_1|\mathbf{Y}] - \beta\mathbf{S}_2,\end{aligned}$$

where the coefficients α and β are given by

$$\alpha \triangleq \frac{\sigma^2\left(\sqrt{\sigma^2-D_1^*}-\rho\sqrt{\sigma^2-D_2^*}\right)}{D_2^*\sqrt{\sigma^2-D_1^*}} \tag{56}$$

$$\beta \triangleq \frac{\sqrt{(\sigma^2-D_1^*)(\sigma^2-D_2^*)}-\rho(\sigma^2-D_2^*)}{D_2^*}, \tag{57}$$

with (D_1^*, D_2^*) as in Assumption C.1. The idea for showing that for this estimator (55) holds, is to exploit the fact that if $\mathbf{Y}$ allows for a "good" estimate $\hat{\mathbf{S}}_1$ of $\mathbf{S}_1$, i.e. if $D_1^* < D_1^{\mathrm{u}}$, then Transmitter 2 can also make a "good" estimate of $\mathbf{W}$, based on $\mathbf{S}_2$ and $\mathbf{Y}$. To show this we first notice that Assumption C.1 implies that there exists a monotonically increasing sequence of integers $\{n_\nu\}$ such that

$$\lim_{\nu\to\infty}\frac{1}{n_\nu}\sum_{k=1}^{n_\nu}\mathsf{E}\left[(S_{i,k}-\hat{S}_{i,k})^2\right] = D_i^* \qquad i\in\{1,2\}. \tag{58}$$

We now derive (55) using the following two lemmas.

Lemma C.2. *For every $\delta > 0$ there exists an $\nu_0(\delta)$ such that for all $\nu > \nu_0(\delta)$ the following inequalities hold*

$$\frac{1}{n_\nu}\sum_{k=1}^{n_\nu}\mathsf{E}\left[S_{1,k}\hat{S}_{1,k}\right] \geq \sigma^2 - D_1^* - \delta, \tag{59}$$

$$\frac{1}{n_\nu}\sum_{k=1}^{n_\nu}\mathsf{E}\left[\hat{S}_{1,k}^2\right] \leq \sigma^2 - D_1^* + \delta, \tag{60}$$

$$\frac{1}{n_\nu}\sum_{k=1}^{n_\nu}\mathsf{E}\left[\hat{S}_{1,k}S_{2,k}\right] \leq \sqrt{(\sigma^2-D_1^*)(\sigma^2-D_2^*)+\delta(\sigma^2+\delta)}. \tag{61}$$

Proof. See Appendix C.4. $\square$

Lemma C.3. *Assumption C.1 and in particular $(D_1^*, D_2^*) \in \mathscr{D}_3$ implies that the coefficients α and β defined in* (56) *and* (57) *satisfy*

$$\alpha \geq 0 \qquad \text{and} \qquad (\rho - \beta) \geq 0. \tag{62}$$

Proof. Follows by noting that for every $(D_1^*, D_2^*) \in \mathscr{D}_3$

$$D_2^* \geq \begin{cases} \left(\sigma^2(1-\rho^2) - D_1^*\right) \frac{\sigma^2}{\sigma^2 - D_1^*} & \text{if } 0 \leq D_1^* \leq \sigma^2(1-\rho^2), \\ \frac{\left(D_1^* - \sigma^2(1-\rho^2)\right)}{\rho^2} & \text{if } D_1^* > \sigma^2(1-\rho^2). \end{cases}$$

$\square$

Using Lemma C.2 and Lemma C.3 we now prove (55) as follows:

$$\begin{aligned}
D_W(\tilde{\varphi}^{(n_\nu)}) &= \frac{1}{n_\nu} \mathsf{E}\left[\|\mathbf{W} - \tilde{\varphi}(\mathbf{S}_2, \mathbf{Y})\|^2\right] \\
&= \frac{1}{n_\nu} \sum_{k=1}^{n_\nu} \mathsf{E}\left[\left(S_{1,k} - \rho S_{2,k} - \alpha \hat{S}_{1,k} + \beta S_{2,k}\right)^2\right] \\
&= \frac{1}{n_\nu} \sum_{k=1}^{n_\nu} \mathsf{E}\left[\left(S_{1,k} - \alpha \hat{S}_{1,k} - (\rho - \beta) S_{2,k}\right)^2\right] \\
&= \frac{1}{n_\nu} \sum_{k=1}^{n_\nu} \Bigg(\mathsf{E}\left[S_{1,k}^2\right] - 2\alpha \mathsf{E}\left[S_{1,k} \hat{S}_{1,k}\right] - 2(\rho - \beta) \mathsf{E}[S_{1,k} S_{2,k}] \\
&\qquad\qquad + \alpha^2 \mathsf{E}\left[\hat{S}_{1,k}^2\right] + 2\alpha(\rho - \beta) \mathsf{E}\left[\hat{S}_{1,k} S_{2,k}\right] \\
&\qquad\qquad + (\rho - \beta)^2 \mathsf{E}\left[S_{2,k}^2\right] \Bigg).
\end{aligned} \tag{63}$$

Using Lemma C.2 and Lemma C.3, as well as $\mathsf{E}[S_{1,k}] = \mathsf{E}[S_{2,k}] = \sigma^2$ and $\mathsf{E}[S_{1,k} S_{2,k}] = \rho\sigma^2$, we now get that for P_1, P_2, N satisfying (17) and for every $\delta > 0$ there exists an $\nu_0(\delta) > 0$ such that for all $\nu > \nu_0(\delta)$,

$$\begin{aligned}
D_W(\tilde{\varphi}^{(n_\nu)}) \leq\ & \sigma^2 - 2\alpha(\sigma^2 - D_1^* - \delta) - 2(\rho - \beta)\rho\sigma^2 \\
& + \alpha^2(\sigma^2 - D_1^* + \delta) + 2\alpha(\rho - \beta)\left(\sqrt{(\sigma^2 - D_1^*)(\sigma^2 - D_2^*) + \delta(\sigma^2 + \delta)}\right) \\
& + (\rho - \beta)^2 \sigma^2.
\end{aligned} \tag{64}$$

Letting ν tend to infinity and then $\delta \to 0$ we obtain from (64) that

$$\begin{aligned}
\varlimsup_{\nu \to \infty} D_W(\tilde{\varphi}^{(n_\nu)}) &\leq \sigma^2 - 2\alpha(\sigma^2 - D_1^*) - 2(\rho - \beta)\rho\sigma^2 \\
&\qquad + \alpha^2(\sigma^2 - D_1^*) + 2\alpha(\rho - \beta)\sqrt{(\sigma^2 - D_1^*)(\sigma^2 - D_2^*)} \\
&\qquad + (\rho - \beta)^2 \sigma^2 \\
&= \sigma^2 \; \frac{2\rho\sqrt{(\sigma^2 - D_1^*)(\sigma^2 - D_2^*)} + D_1^* + D_2^* - \sigma^2(1 + \rho^2)}{D_2^*},
\end{aligned} \tag{65}$$

where in the last step we have replaced the terms α and β by their expressions in (56) and (57). To conclude our upper bound we now make use of one last lemma.

Lemma C.4. *For all* $(D_1^*, D_2^*) \in \operatorname{int}(\mathscr{D}_3)$ *the expression on the RHS of* (65) *is strictly increasing in* D_1^*.

Proof. Denote by $\tilde{D}_W$ the RHS of (65). The proof follows by showing that for all $(D_1^*, D_2^*) \in \operatorname{int}(\mathscr{D}_3)$

$$\frac{\partial \tilde{D}_W}{\partial D_1^*} > 0.$$

This follows by direct differentiation and by noting that for $(D_1^*, D_2^*) \in \operatorname{int}(\mathscr{D}_3)$

$$D_2^* > \begin{cases} \left(\sigma^2(1-\rho^2) - D_1^*\right) \frac{\sigma^2}{\sigma^2 - D_1^*} & \text{if } 0 \leq D_1^* \leq \sigma^2(1-\rho^2), \\ \frac{\left(D_1^* - \sigma^2(1-\rho^2)\right)}{\rho^2} & \text{if } D_1^* > \sigma^2(1-\rho^2). \end{cases}$$

□

Since $D_1^* < D_1^{\mathrm{u}}$ and $D_2^* = D_2^{\mathrm{u}}$ it follows from (65) and Lemma C.4 that

$$\begin{aligned} \varlimsup_{\nu \to \infty} D_W(\tilde{\varphi}^{(n_\nu)}) &< \sigma^2 \, \frac{2\rho\sqrt{(\sigma^2 - D_1^{\mathrm{u}})(\sigma^2 - D_2^{\mathrm{u}})} + D_1^{\mathrm{u}} + D_2^{\mathrm{u}} - \sigma^2(1+\rho^2)}{D_2^{\mathrm{u}}} \\ &= \sigma^2 \frac{N(1-\rho^2)}{P_1(1-\rho^2) + N}, \end{aligned} \tag{66}$$

where the last line follows from replacing D_1^{u} and D_2^{u} by their expressions given in Theorem 3.6. Thus, we have proven (55).

C.3 Concluding the Proof of Theorem 3.6

It follows from (48) and (55) that Assumption C.1 is false. We now show that this implies that if P_1, P_2, N satisfy (17) with strict inequality, then no pair (D_1, D_2) satisfying $D_1 < D_1^{\mathrm{u}}$ and $D_2 \leq D_2^{\mathrm{u}}$ or satisfying $D_1 \leq D_1^{\mathrm{u}}$ and $D_2 < D_2^{\mathrm{u}}$ is achievable. To prove this we assume $\rho > 0$ because for $\rho = 0$ Condition (17) becomes $P_1 P_2 \leq 0$ and is therefore never satisfied with strict inequality.

Our arguments are given in the following sequence of statements:

A) If P_1, P_2, N satisfy (17) with strict inequality, then the set of (D_1^*, D_2^*) satisfying (47) is not empty.

Statement A) holds since if P_1, P_2, N satisfy (17) with strict inequality, then $(D_1^{\mathrm{u}}, D_2^{\mathrm{u}}) \in \operatorname{int}(\mathscr{D}_3)$ and $\operatorname{int}(\mathscr{D}_3) \neq \emptyset$ whenever $\rho \neq 0$.

B) If P_1, P_2, N satisfy (17) with strict inequality, then there do not exist encoding rules, that, when combined with the optimal conditional expectation reconstructors, result in (D_1^*, D_2^*) as defined in (46) satisfying

$$D_1^* < D_1^{\mathrm{u}} \qquad \text{and} \qquad D_2^* = D_2^{\mathrm{u}},$$

(with (D_1^*, D_2^*) in or outside $\operatorname{int}(\mathscr{D}_3)$).

Statement B) can be shown by contradiction. If a coding scheme as described in B) were to exist, then by time-sharing it with the uncoded scheme—for which $(D_1^{\mathrm{u}}, D_2^{\mathrm{u}}) \in \operatorname{int}(\mathscr{D}_3)$—and by Statement A), we would obtain a scheme for which (D_1^*, D_2^*) satisfies (47), in contradiction to the fact that Assumption C.1 is false.

C) If P_1, P_2, N satisfy (17) with strict inequality, then there exist no encoding rules, which, when combined with the optimal conditional expectation reconstructors, result in (D_1^*, D_2^*) as defined in (46) such that

$$D_1^* = D_1^{\mathrm{u}} \qquad \text{and} \qquad D_2^* < D_2^{\mathrm{u}}.$$

Statement C) can be proved using arguments similar to those used to prove Statement B).

D) If P_1, P_2, N satisfy (17) with strict inequality, then there exist no encoding rules, which when combined with the optimal conditional expectation reconstructors, result in (D_1^*, D_2^*) as defined in (46) such that $D_1^* < D_1^{\mathrm{u}}$ and $D_2^* \leq D_2^{\mathrm{u}}$ or such that $D_1^* \leq D_1^{\mathrm{u}}$ and $D_2^* < D_2^{\mathrm{u}}$.

To show Statement D) we proceed by contradiction. To this end, consider two variations of our uncoded scheme. Call these two variations "Scheme U_1" and "Scheme U_2". Let Scheme U_1 be given by the channel inputs

$$X_{1,k}^{\mathrm{u}_1} = \sqrt{\frac{P_1}{\sigma^2}} S_{1,k} \qquad \text{and} \qquad X_{2,k}^{\mathrm{u}_1} = 0,$$

and the optimal conditional expectation reconstructors $\hat{\mathbf{S}}_1 = \mathsf{E}\left[\mathbf{S}_1 \middle| \mathbf{Y}\right]$ and $\hat{\mathbf{S}}_2 = \mathsf{E}\left[\mathbf{S}_2 \middle| \mathbf{Y}\right]$. The resulting distorion pair $(D_1^{\mathrm{u}_1}, D_2^{\mathrm{u}_1})$ is given by

$$D_1^{\mathrm{u}_1} = \sigma^2 \frac{N}{P_1 + N} \qquad\qquad D_2^{\mathrm{u}_1} = \sigma^2 \frac{(1-\rho^2)P_1 + N}{P_1 + N}.$$

Similarly, let Scheme U_2 be given by the channel inputs

$$X_{1,k}^{\mathrm{u}_2} = 0 \qquad \text{and} \qquad X_{2,k}^{\mathrm{u}_2} = \sqrt{\frac{P_2}{\sigma^2}} S_{2,k},$$

and the same optimal conditional expectation reconstructors as for Scheme U_1. The resulting distorion pair $(D_1^{\mathrm{u}_2}, D_2^{\mathrm{u}_2})$ is given by

$$D_1^{\mathrm{u}_2} = \sigma^2 \frac{(1-\rho^2)P_2 + N}{P_2 + N} \qquad\qquad D_2^{\mathrm{u}_2} = \sigma^2 \frac{N}{P_2 + N}.$$

Now assume there would exist a coding scheme as described in D). Since $D_2^{\mathrm{u}_1} > D_2^{\mathrm{u}}$ and $D_1^{\mathrm{u}_2} > D_1^{\mathrm{u}}$ it would follow from time-sharing either with Scheme U_1 or Scheme U_2 that Statement B) or Statement C) is false.

E) If P_1, P_2, N satisfy (17) with strict inequality, then there exist no coding scheme resulting in (D_1^*, D_2^*) as defined in (46) such that

$$D_1^* < D_1^{\mathrm{u}} \qquad \text{and} \qquad D_2^* \leq D_2^{\mathrm{u}},$$

(be the reconstruction rule optimal or not).

Statement E) follows from D) because no reconstructor $\phi_i^{(n)}$ can outperform the optimal conditional expectation reconstructor $\hat{\mathbf{S}}_i = \mathsf{E}\left[\mathbf{S}_i \middle| \mathbf{Y}\right]$.

By Statement E) it follows that if P_1, P_2, N satisfy (17) with strict inequality, then no (D_1, D_2) satisfying $D_1 < D_1^{\mathrm{u}}$ and $D_2 \leq D_2^{\mathrm{u}}$ is achievable.

C.4 Proof of Lemma C.2

By (58) it follows that for every $\delta > 0$ there exists a $\nu_0(\delta) > 0$ such that for all $\nu > \nu_0(\delta)$

$$D_i^* - \delta < \frac{1}{n_\nu} \sum_{k=1}^{n_\nu} \mathsf{E}\left[(S_{i,k} - \hat{S}_{i,k})^2\right] < D_i^* + \delta \qquad i \in \{1, 2\}. \tag{67}$$

Using (67), the relation $\mathsf{E}\left[S_{1,k}^2\right] = \sigma^2$, and (45) we obtain that

$$\sigma^2 - D_1^* - \delta \leq \frac{1}{n_\nu} \sum_{k=1}^{n_\nu} \mathsf{E}\left[S_{1,k} \hat{S}_{1,k}\right] \leq \sigma^2 - D_1^* + \delta, \tag{68}$$

and that

$$\sigma^2 - D_1^* - \delta \leq \frac{1}{n_\nu} \sum_{k=1}^{n_\nu} \mathsf{E}\left[\hat{S}_{1,k}^2\right] \leq \sigma^2 - D_1^* + \delta. \tag{69}$$

This proves Inequalities (59) and (60).

To prove (61) we note that for every $c \in \mathbb{R}$ we can view $c\hat{S}_{1,k}$ as an estimator of $S_{2,k}$ based on $\mathbf{Y}$. As such it cannot outperform the optimal estimator of $S_{2,k}$ given by $\mathbf{Y}$, namely the estimator $\hat{\mathbf{S}}_2 = \mathsf{E}[\mathbf{S}_2|\mathbf{Y}]$. Consequently, for every $\delta > 0$ it follows by (67) that there exists an $\nu_0(\delta) > 0$ such that for all $\nu > \nu_0(\delta)$ and all $c \in \mathbb{R}$,

$$\begin{aligned} \frac{1}{n_\nu} \sum_{k=1}^{n_\nu} \mathsf{E}\left[(S_{2,k} - c\hat{S}_{1,k})^2\right] &\geq \frac{1}{n_\nu} \sum_{k=1}^{n_\nu} \mathsf{E}\left[(S_{2,k} - \hat{S}_{1,k})^2\right] \\ &> D_2^* - \delta. \end{aligned} \tag{70}$$

Rewriting (70) gives

$$\sigma^2 - 2c \frac{1}{n_\nu} \sum_{k=1}^{n_\nu} \mathsf{E}\left[S_{2,k} \hat{S}_{1,k}\right] + c^2(\sigma^2 - D_1^* + \delta) > D_2^* - \delta,$$

and choosing

$$c = \sqrt{\frac{\sigma^2 - D_2^* - \delta}{\sigma^2 - D_1^* + \delta}},$$

yields that for all $\nu > \nu_0(\delta)$

$$\begin{aligned} \frac{1}{n_\nu} \sum_{k=1}^{n_\nu} \mathsf{E}\left[S_{2,k} \hat{S}_{1,k}\right] &\leq \sqrt{(\sigma^2 - D_1^* + \delta)(\sigma^2 - D_2^* - \delta)} \\ &= \sqrt{(\sigma^2 - D_1^*)(\sigma^2 - D_2^*) - \delta(D_2^* - D_1^* + \delta)} \\ &\leq \sqrt{(\sigma^2 - D_1^*)(\sigma^2 - D_2^*) + \delta(\sigma^2 + \delta)}. \end{aligned}$$

□

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